Distributional Sources for Newman's Holomorphic Field

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Abstract

In [N73], Newman considered the holomorphic extension $\tilde{\boldsymbol{E}}(\boldsymbol{z})$ of the Coulomb field $\boldsymbol{E}(\boldsymbol{x})$ in $\mathbb{R}^3$. By analyzing its multipole expansion, he showed that the real and imaginary parts

$$\boldsymbol{E}(\boldsymbol{x}+i\boldsymbol{y}) \equiv \operatorname{Re} \tilde{\boldsymbol{E}}(\boldsymbol{x}+i\boldsymbol{y}), \quad \boldsymbol{H}(\boldsymbol{x}+i\boldsymbol{y}) \equiv \operatorname{Im} \tilde{\boldsymbol{E}}(\boldsymbol{x}+i\boldsymbol{y}),$$

viewed as functions of $\boldsymbol{x}$, are the electric and magnetic fields generated by a *spinning ring of charge* $\mathcal{R}$. This represents the EM part of the Kerr-Newman solution to the Einstein-Maxwell equations [NJ65, N65]. As already pointed out in [NJ65], this interpretation is somewhat problematic since the fields are double-valued. To make them single-valued, a branch cut must be introduced so that $\mathcal{R}$ is replaced by a *charged disk* $\mathcal{D}$ having $\mathcal{R}$ as its boundary. In the context of curved spacetime, $\mathcal{D}$ becomes a *spinning disk of charge and mass* representing the singularity of the Kerr-Newman solution.

Here we confirm the above interpretation of $\boldsymbol{E}$ and $\boldsymbol{H}$ without resorting to asymptotic expansions, by computing the charge- and current densities *directly* as distributions in $\mathbb{R}^3$ supported in $\mathcal{D}$. This shows that $\mathcal{D}$ *spins rigidly at the critical rate so that its rim* $\mathcal{R}$ *moves at the speed of light.*

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1. Introduction

The holomorphic extension of the Euclidean distance $r(\boldsymbol{x})$ in $\mathbb{R}^n$ was studied in [K00]. Here we recall the basics for $n = 3$. Let

$$\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} \in \mathbb{C}^3, \quad |\boldsymbol{x}| = r, \quad |\boldsymbol{y}| = a.$$

Define the *complex distance* in $\mathbb{C}^3$ as

$$\tilde{r}(\boldsymbol{z}) = \sqrt{\boldsymbol{z} \cdot \boldsymbol{z}} = \sqrt{r^2 - a^2 + 2i\boldsymbol{x} \cdot \boldsymbol{y}} \equiv p + iq,$$

whose branch points, for given $\boldsymbol{y} \in \mathbb{R}^3$, form a *ring* in $\mathbb{R}^3$:

$$\mathcal{R}(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}^3 : \tilde{r} = 0\} = \{\boldsymbol{x} \in \mathbb{R}^3 : r = a, \ \boldsymbol{x} \cdot \boldsymbol{y} = 0\}.$$

To make $\tilde{r}$ single-valued, choose the branch with $p \geq 0$, so that $\tilde{r}(\boldsymbol{x}) = +r$. It then follows that

$$0 \leq p \leq r, \qquad -a \leq q \leq a,$$

and q has a discontinuity across the *branch disk*

$$\mathcal{D}(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}^3 : p = 0\} = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x} \cdot \boldsymbol{y} = 0, \ r \leq a\}$$

given by

$$\boldsymbol{x} \to \mathcal{D}^{\pm}(\boldsymbol{y}) \equiv \mathcal{D}(\boldsymbol{y}) \pm i0\boldsymbol{y} \ \Rightarrow \ \tilde{r} \to \pm i\sqrt{a^2 - \varrho^2}, \tag{1}$$

where ϱ is the radial variable on $\mathcal{D}$.

$\mathcal{D}$ can be deformed continuously to *any* surface with $\partial \mathcal{D} = \mathcal{R}$, so it is really a *flexible membrane spanning* $\mathcal{R}$.

The holomorphic Coulomb potential of a unit charge and its field are defined by

$$\tilde{\phi}(\boldsymbol{z}) = \frac{1}{4\pi\tilde{r}}, \quad \tilde{\boldsymbol{E}}(\boldsymbol{z}) = -\nabla\tilde{\phi} = \frac{\boldsymbol{z}}{4\pi\tilde{r}^3}. \tag{2}$$

Because $\tilde{\phi}$ is discontinuous across $\mathcal{D}$, we specify that ∇ be the *distributional gradient with respect to* $\boldsymbol{x}$.

Intuitively, think of $\tilde{\phi}(\boldsymbol{x} + i\boldsymbol{y})$ and $\tilde{\boldsymbol{E}}(\boldsymbol{x} + i\boldsymbol{y})$ as the Coulomb potential and Coulomb field at $\boldsymbol{x}$ of a unit charge displaced from the origin to the imaginary location $-i\boldsymbol{y}$.

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We now show that this view is physically viable and leads to some interesting and unexpected consequences.

As in [N73, NW74, NW74a] (see also [K94, Chapter 9]), define the real fields $\boldsymbol{E}, \boldsymbol{H}$ on $\mathbb{C}^3$ by

$$\tilde{\boldsymbol{E}}(\boldsymbol{z}) = \boldsymbol{E}(\boldsymbol{z}) + i\boldsymbol{H}(\boldsymbol{z}).$$

Regarding these as electric and magnetic fields, we will compute their source distributions.

The *inhomogeneous* Maxwell equations state that the charge and current densities are

$$\rho(\boldsymbol{z}) = \nabla \cdot \tilde{\boldsymbol{E}}(\boldsymbol{z}) = -\Delta\tilde{\phi}, \quad \boldsymbol{J}(\boldsymbol{z}) = -i\nabla \times \tilde{\boldsymbol{E}}(\boldsymbol{z}), \tag{3}$$

while the *homogeneous* Maxwell equations require that ρ and $\boldsymbol{J}$ be *real.* The divergence and curl are the distributional operators.

As $\tilde{\phi}$ and $\tilde{\boldsymbol{E}}$ are holomorphic when $\boldsymbol{x} \notin \mathcal{D}(\boldsymbol{y})$, we may compute ρ and $\boldsymbol{J}$ outside of $\mathcal{D}$ by ordinary differentiation. Since $\tilde{\phi}$ is harmonic and $\tilde{\boldsymbol{E}}$ is a gradient, this gives

$$\rho(\boldsymbol{x} + i\boldsymbol{y}) = 0, \ \ \boldsymbol{J}(\boldsymbol{x} + i\boldsymbol{y}) = \boldsymbol{0} \quad \text{for all} \quad \boldsymbol{x} \notin \mathcal{D}(\boldsymbol{y}).$$

The main goal of this paper is to compute ρ and $\boldsymbol{J}$ as distributions in $\boldsymbol{x}$ supported on $\mathcal{D}(\boldsymbol{y})$. This will be done in two stages:

• First compute the *surface* charge- and current densities on the *interior* of $\mathcal{D}$. These will be seen to diverge on the rim $\mathcal{R}$.

• Then compute the *volume* densities $\rho(\boldsymbol{x} + i\boldsymbol{y})$ and $\boldsymbol{J}(\boldsymbol{x} + i\boldsymbol{y})$ as distributions in $\boldsymbol{x} \in \mathbb{R}^3$, for any fixed $\boldsymbol{y} \in \mathbb{R}^3$.

Of course the results of the second computation include those of the first, but the first stage adds some clarity and makes contact with the usual method of bounday conditions.

2. Interior Surface Densities

As $\boldsymbol{x} \to \mathcal{D}^{\pm}(\boldsymbol{y})$, (1) states that $\tilde{r} \to \pm i\sqrt{a^2 - \varrho^2}$, thus

$$\tilde{\boldsymbol{E}} = \frac{\boldsymbol{x} + i\boldsymbol{y}}{4\pi\tilde{r}^3} \to \pm i\frac{\boldsymbol{x} + i\boldsymbol{y}}{4\pi(a^2 - \varrho^2)^{3/2}},$$

The fields on $\mathcal{D}^{\pm}$ are therefore

$$\boldsymbol{E}^{\pm} = \frac{\mp\boldsymbol{y}}{4\pi(a^2 - \varrho^2)^{3/2}}, \quad \boldsymbol{H}^{\pm} = \frac{\pm\boldsymbol{x}}{4\pi(a^2 - \varrho^2)^{3/2}},$$

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with $\boldsymbol{E}^{\pm}$ normal to $\mathcal{D}$ and $\boldsymbol{H}^{\pm}$ parallel to $\mathcal{D}$. Their jumps are

$$\delta\boldsymbol{E} \equiv \boldsymbol{E}^{+} - \boldsymbol{E}^{-} = -\frac{\boldsymbol{y}}{2\pi(a^2 - \varrho^2)^{3/2}}$$

$$\delta\boldsymbol{H} \equiv \boldsymbol{H}^{+} - \boldsymbol{H}^{-} = \frac{\boldsymbol{x}}{2\pi(a^2 - \varrho^2)^{3/2}}\,.$$

By placing infinitesimal pillboxes and loops around interior points of $\mathcal{D}(\boldsymbol{y})$ as in [J99, p. 17], we draw the following conclusions:

• Since the tangential component of $\boldsymbol{E}$ and the normal component of $\boldsymbol{H}$ are continuous across $\mathcal{D}$ (they actually vanish on $\mathcal{D}^{\pm}$), the *homogeneous* Maxwell equations are satisfied everywhere except possibly on $\mathcal{R}$ (where $\boldsymbol{E}^{\pm}, \boldsymbol{H}^{\pm}$ diverge):

$$\nabla \times \boldsymbol{E}(\boldsymbol{x} + i\boldsymbol{y}) = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{H}(\boldsymbol{x} + i\boldsymbol{y}) = 0 \quad \forall \boldsymbol{x} \notin \mathcal{R}.$$

(We will see that these equations also hold on $\mathcal{R}$.)

• The surface charge- and current densities on $\mathcal{D}$ are

$$\sigma = \hat{\boldsymbol{y}} \cdot \delta\boldsymbol{E} = -\frac{a}{2\pi(a^2 - \varrho^2)^{3/2}}\,, \qquad \hat{\boldsymbol{y}} \equiv \boldsymbol{y}/a \tag{4}$$

$$\boldsymbol{K} = \hat{\boldsymbol{y}} \times \delta\boldsymbol{H} = \frac{\hat{\boldsymbol{y}} \times \boldsymbol{x}}{2\pi(a^2 - \varrho^2)^{3/2}} = \frac{\varrho\,\boldsymbol{e}_{\varphi}}{2\pi(a^2 - \varrho^2)^{3/2}}\,, \tag{5}$$

where $\boldsymbol{e}_{\varphi}$ is the unit vector along the azimuthal coordinate φ.

• The *velocity* of the charge at (ϱ, z, φ) is

$$\boldsymbol{v} = \frac{c\boldsymbol{K}}{\sigma} = -\frac{c\varrho\,\boldsymbol{e}_{\varphi}}{a}\,, \tag{6}$$

where we have inserted the speed of light c.

• The charge moves at a constant angular velocity $\boldsymbol{\omega} = -\hat{\boldsymbol{y}}c/a$.

• Displacing the point charge to $+i\boldsymbol{y}$ would thus give

$$\boxed{\boldsymbol{\omega} = \hat{\boldsymbol{y}}\,c/a.} \tag{7}$$

• *This rigid rotational motion is the maximum allowed by relativity, since* $|\boldsymbol{v}| \to c$ *on the rim* $\mathcal{R}$.

• This is related to the singularity on $\mathcal{R}$, since (6) implies that

$$\sigma = -\frac{1}{2\pi a^2(1 - v^2/c^2)^{3/2}}\,, \quad \boldsymbol{K} = \frac{\varrho\,\boldsymbol{e}_{\varphi}}{2\pi a^3(1 - v^2/c^2)^{3/2}}\,. \tag{8}$$

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However, the above surface densities cannot be the entire story, as can be seen from the following considerations.

• It is difficult to see how the surface densities $\sigma, \boldsymbol{K}$ are related to the distributions for a point source at the origin ($\boldsymbol{y} = \boldsymbol{0}$), which ought to be $\rho(\boldsymbol{x}) = \delta(\boldsymbol{x})$, $\boldsymbol{J}(\boldsymbol{x}) = \boldsymbol{0}$.

• In particular, according to (4) the total charge with *any* choice of $\boldsymbol{y}$ would not be $e = 1$, as we have assumed, but

$$e(\boldsymbol{y}) = \int_{\mathcal{D}(\boldsymbol{y})} \sigma(\boldsymbol{x} + i\boldsymbol{y}) \varrho \, d\varrho \, d\varphi = 2\pi \int_0^a \sigma(\varrho) \, \varrho \, d\varrho = -\infty.$$

This shows that σ and $\boldsymbol{K}$ are insufficient, and we must also compute the *singular* parts of the densities supported on $\mathcal{R}$.

3. The Volume Charge Distribution

We first compute the charge density ρ as a distribution on $\mathbb{R}^3$, using the *regularization* method introduced in [K00].

Given $\varepsilon > 0$, the set

$$\mathcal{D}_\varepsilon(\boldsymbol{y}) \equiv \{\boldsymbol{x} \in \mathbb{R}^3 : p = \varepsilon\}$$

is an ellipsoid enclosing $\mathcal{D}$, given in cylindrical coordinates by

$$\frac{\varrho^2}{a^2 + \varepsilon^2} + \frac{z^2}{\varepsilon^2} = 1.$$

Define the *truncated field* by

$$\tilde{\boldsymbol{E}}_\varepsilon(\boldsymbol{z}) = \theta(p - \varepsilon) \tilde{\boldsymbol{E}}(\boldsymbol{z}), \tag{9}$$

where θ is the Heaviside step function. Since ρ and $\boldsymbol{J}$ vanish outside of $\mathcal{D}$, we have

$$\theta(p - \varepsilon) \nabla \cdot \tilde{\boldsymbol{E}} \equiv 0, \quad \theta(p - \varepsilon) \nabla \times \tilde{\boldsymbol{E}} \equiv \boldsymbol{0}.$$

Define the *ε-equivalent charge- and current distributions* by

$$\rho_\varepsilon(\boldsymbol{z}) \equiv \nabla \cdot \tilde{\boldsymbol{E}}_\varepsilon = \delta(p - \varepsilon) \nabla p \cdot \tilde{\boldsymbol{E}}$$
$$\boldsymbol{J}_\varepsilon(\boldsymbol{z}) \equiv -i\nabla \times \tilde{\boldsymbol{E}}_\varepsilon = -i\delta(p - \varepsilon) \nabla p \times \tilde{\boldsymbol{E}}.$$

These are the distributions on $\mathcal{D}_\varepsilon(\boldsymbol{y})$ needed to give exactly the same field $\tilde{\boldsymbol{E}}$ *outide* $\mathcal{D}_\varepsilon$ while giving a vanishing field *inside* $\mathcal{D}_\varepsilon$.

Since $\mathcal{D}_\varepsilon$ has no boundary, we expect ρ_ε and $\boldsymbol{J}_\varepsilon$ to be representable by smooth *surface* densities on $\mathcal{D}_\varepsilon$.

From [K00], we have (with $\tilde{r}^*$ the complex conjugate of $\tilde{r}$)

$$\nabla p \cdot \tilde{\boldsymbol{E}} = \frac{p\boldsymbol{x} + q\boldsymbol{y}}{\tilde{r}^*\tilde{r}} \cdot \frac{\boldsymbol{x} + i\boldsymbol{y}}{4\pi\tilde{r}^3} = \frac{p(r^2 + q^2) + iq(a^2 + p^2)}{4\pi\tilde{r}^*\tilde{r}^4}$$

$$= \frac{(p + iq)(a^2 + p^2)}{4\pi\tilde{r}^*\tilde{r}^4} = \frac{a^2 + p^2}{4\pi\tilde{r}^*\tilde{r}^4}$$

$$\nabla p \times \tilde{\boldsymbol{E}} = \frac{p\boldsymbol{x} + q\boldsymbol{y}}{\tilde{r}^*\tilde{r}} \times \frac{\boldsymbol{x} + i\boldsymbol{y}}{4\pi\tilde{r}^3} = \frac{ip - q}{4\pi\tilde{r}^*\tilde{r}^4}\,\boldsymbol{x} \times \boldsymbol{y} = i\frac{\boldsymbol{x} \times \boldsymbol{y}}{4\pi\tilde{r}^*\tilde{r}^4}\,.$$

This gives (with $\tilde{r} = \varepsilon + iq$)

$$\rho_\varepsilon = \delta(p - \varepsilon)\frac{a^2 + \varepsilon^2}{4\pi\tilde{r}^*\tilde{r}^3}, \qquad \boldsymbol{J}_\varepsilon = \delta(p - \varepsilon)\frac{\boldsymbol{x} \times \boldsymbol{y}}{4\pi\tilde{r}^*\tilde{r}^3}. \tag{10}$$

The ε-equivalent sources are *complex*. Their imaginary parts are *fictitious magnetic sources* introduced by making the normal component of $\boldsymbol{H}$ and the tangential component of $\boldsymbol{E}$ discontinuous across the ellipsoid $\mathcal{D}_\varepsilon$.

The *true* sources are then defined as

$$\rho(\boldsymbol{x} + i\boldsymbol{y}) \equiv \lim_{\varepsilon \to 0} \rho_\varepsilon(\boldsymbol{x} + i\boldsymbol{y}), \quad \boldsymbol{J}(\boldsymbol{x} + i\boldsymbol{y}) \equiv \lim_{\varepsilon \to 0} \boldsymbol{J}_\varepsilon(\boldsymbol{x} + i\boldsymbol{y}),$$

where the limit is in the sense of distributions in $\boldsymbol{x}$, for given $\boldsymbol{y}$.

It will be found that ρ and $\boldsymbol{J}$ are *real* distributions, in agreement with the earlier observation that the *homogeneous* Maxwell equations are satisfied.

As shown in [K00], (p, q, φ) are *oblate spheroidal coordinates* in $\mathbb{R}^3$. The level sets of p are oblate spheroids $\mathcal{D}_p$ like $\mathcal{D}_\varepsilon$, and those of q are hyperboloids $\mathcal{H}_q$ orthogonal to $\mathcal{D}_p$.

A test function in $\mathbb{R}^3$ can be expressed as

$$f(\boldsymbol{x}) = f^\sharp(p, q, \varphi), \qquad p \geq 0, \ -a \leq q \leq a, \ 0 \leq \varphi \leq 2\pi.$$

In terms of (p, q, φ), the volume measure in $\mathbb{R}^3$ is

$$d\boldsymbol{x} = \frac{1}{a}\,\tilde{r}^*\tilde{r}\,dp\,dq\,d\varphi, \tag{11}$$

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so the distributional action of ρ_ε on f is

$$
\begin{aligned}
\langle\, \rho_\varepsilon\,, f \,\rangle &\equiv \int_{\mathbb{R}^3} d\boldsymbol{x}\; \rho_\varepsilon(\boldsymbol{x} + i\boldsymbol{y})\, f(\boldsymbol{x}) \\
&= \frac{1}{a}\int_0^\infty dp \int_{-a}^a dq\, \tilde{r}^*\tilde{r}\, \rho_\varepsilon^\sharp(p,q) \int_0^{2\pi} d\varphi\; f^\sharp(p,q,\varphi) \\
&= \frac{2\pi}{a}\int_0^\infty \tilde{r}^*\tilde{r}\, dp \int_{-a}^a dq\, \rho_\varepsilon^\sharp(p,q)\bar{f}^\sharp(p,q),
\end{aligned}
$$

where $\bar{f}^\sharp(p,q)$ is the *mean* of $f^\sharp(p,q,\varphi)$ over φ.

Inserting (10) now gives

$$
\langle\, \rho_\varepsilon\,, f \,\rangle = \frac{a^2 + \varepsilon^2}{2a}\int_{-a}^a dq\, \frac{\bar{f}^\sharp(\varepsilon,q)}{(\varepsilon + iq)^2}\,.
$$

As claimed above, ρ_ε is represented by a smooth (but complex) surface density on $\mathcal{D}_\varepsilon$.

To obtain a finite limit as $\varepsilon \to 0$, subtract and add the linear Taylor polynomial $\bar{f}^\sharp(\varepsilon,0) + q\partial_q\bar{f}^\sharp(\varepsilon,0)$ in the numerator:

$$
\begin{aligned}
\langle\, \rho_\varepsilon\,, f \,\rangle = {} & \frac{a^2 + \varepsilon^2}{2a}\int_{-a}^a dq\, \frac{\bar{f}^\sharp(\varepsilon,q) - \bar{f}^\sharp(\varepsilon,0) - q\partial_q\bar{f}^\sharp(\varepsilon,0)}{(\varepsilon + iq)^2} \\
& + \frac{a^2 + \varepsilon^2}{2a}\int_{-a}^a dq\, \frac{\bar{f}^\sharp(\varepsilon,0) + q\partial_q\bar{f}^\sharp(\varepsilon,0)}{(\varepsilon + iq)^2}.
\end{aligned}
$$

In Section 4 of [K00], we found that

$$
\int_{-a}^a \frac{dq}{(\varepsilon + iq)^2} = \frac{2a}{a^2 + \varepsilon^2} \tag{12}
$$

$$
\int_{-a}^a \frac{iq\, dq}{(\varepsilon + iq)^2} = \pi - 2\tan^{-1}(\varepsilon/a) - \frac{2\varepsilon a}{a^2 + \varepsilon^2}\,. \tag{13}
$$

Inserting these into the above expression and taking the limit $\varepsilon \to 0$ now gives the action of the *true* charge distribution as

$$
\begin{aligned}
\langle\, \rho\,, f \,\rangle = {} & -\frac{a}{2}\int_{-a}^a dq\, \frac{\bar{f}^\sharp(0,q) - \bar{f}^\sharp(0,0) - q\partial_q\bar{f}^\sharp(0,0)}{q^2} \\
& + \bar{f}^\sharp(0,0) - i\frac{\pi a}{2}\partial_q\bar{f}^\sharp(0,0).
\end{aligned}
$$

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Since the test function is continuously differentiable and $(0, q, \varphi)$ and $(0, -q, \varphi)$ represent the same point on $\mathcal{D}$, we have

$$\bar{f}^\sharp(0, -q) = \bar{f}^\sharp(0, q) \quad \text{and} \quad \partial_q \bar{f}^\sharp(0, 0) = 0,$$

and

$$\langle \rho, f \rangle = \bar{f}^\sharp(0, 0) - a \int_0^a dq \, \frac{\bar{f}^\sharp(0, q) - \bar{f}^\sharp(0, 0)}{q^2}. \tag{14}$$

The cylindrical coordinates (ϱ, z) are related to (p, q) by

$$z \equiv \boldsymbol{x} \cdot \hat{\boldsymbol{y}} = \frac{pq}{a}, \quad \varrho \equiv \sqrt{r^2 - z^2} = \frac{\sqrt{a^2 + p^2}\sqrt{a^2 - q^2}}{a}. \tag{15}$$

Writing

$$f^\flat(\varrho, z, \varphi) \equiv f^\sharp(p, q, \varphi), \quad \bar{f}^\flat(\varrho, z) \equiv \bar{f}^\sharp(p, q),$$

we obtain the action of ρ in cylindrical coordinates as

$$\boxed{\langle \rho, f \rangle = \bar{f}^\flat(a, 0) - a \int_0^a \varrho \, d\varrho \, \frac{\bar{f}^\flat(\varrho, 0) - \bar{f}^\flat(a, 0)}{(a^2 - \varrho^2)^{3/2}}.} \tag{16}$$

The first term is the mean of f on $\mathcal{R}$, while the second term is a *regularization* of the surface density σ in (4), taking into account the singularity on $\mathcal{R}$.

  • Note that ρ is *real*, so there are *no magnetic charges:* $\nabla \cdot \boldsymbol{H} \equiv 0$.

  • The subtraction in the second term makes ρ a *regularized distribution* in the sense of [GS64], of the same type as the Cauchy principal value integral. It is *not* an ordinary function to which values can be assigned at points.

  • However, if $f(\boldsymbol{x}) = 0$ on $\mathcal{R}$, then (16) becomes

$$\langle \rho, f \rangle = -\frac{a}{2\pi} \int_0^a \frac{\varrho \, d\varrho}{(a^2 - \varrho^2)^{3/2}} \int_0^{2\pi} f^\flat(\varrho, 0, \varphi) \, d\varphi,$$

reproducing the surface charge density σ in (4).

  • For a *constant* test function, the second term in (16) vanishes. This term therefore represents a *single layer of zero net charge.*

  • Taking $f(\boldsymbol{x}) \equiv 1$ gives the correct total charge as promised:

$$e(\boldsymbol{y}) = \int_{\mathbb{R}^3} \rho(\boldsymbol{x} + i\boldsymbol{y}) \, d\boldsymbol{x} = \langle \rho, 1 \rangle = 1 \qquad \forall \boldsymbol{y} \in \mathbb{R}^3.$$

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- Intuitively, therefore, the negative-infinite charge of σ is balanced by a positive-infinite charge on the rim, just so as to give the correct total charge $e = 1$.

- Unlike the singular expression (4) for σ, the distribution (16) reduces to the point source as $\boldsymbol{y} \to \boldsymbol{0}$:

$$\langle\, \rho\,,f\,\rangle \to \bar{f}^{\flat}(0,0) = f(\boldsymbol{0}), \quad \text{hence} \quad \rho(\boldsymbol{x} + i\boldsymbol{y}) \to \delta(\boldsymbol{x}).$$

4. The Volume Current Distribution

Writing

$$\boldsymbol{x} = \boldsymbol{\varrho} + z\hat{\boldsymbol{y}} = \varrho\,\boldsymbol{e}_{\varrho} + z\hat{\boldsymbol{y}} \tag{17}$$

gives

$$\boldsymbol{J}_{\varepsilon} = \delta(p - \varepsilon)\,\frac{\boldsymbol{x} \times \boldsymbol{y}}{4\pi\tilde{r}^{*}\tilde{r}^{3}} = -\delta(p - \varepsilon)\,\frac{a\varrho\,\boldsymbol{e}_{\varphi}}{4\pi\tilde{r}^{*}\tilde{r}^{3}}\,. \tag{18}$$

Let $\boldsymbol{f}(\boldsymbol{x})$ be a vector-valued test function in $\mathbb{R}^{3}$. Then

$$\boldsymbol{J}_{\varepsilon} \cdot \boldsymbol{f} = -a\delta(p - \varepsilon)\,\frac{h^{\sharp}(p,q,\varphi)}{4\pi\tilde{r}^{*}\tilde{r}^{3}},$$

where

$$h^{\sharp}(p,q,\varphi) \equiv \varrho\,\boldsymbol{e}_{\varphi} \cdot \boldsymbol{f}^{\sharp}(p,q,\varphi) \equiv \varrho f_{\varphi}^{\sharp}(p,q,\varphi).$$

The distributional action of $\boldsymbol{J}_{\varepsilon}$ on $\boldsymbol{f}$ is therefore

$$\langle\, \boldsymbol{J}_{\varepsilon}\,,\boldsymbol{f}\,\rangle \equiv \int_{\mathbb{R}^{3}} d\boldsymbol{x}\,\boldsymbol{J}_{\varepsilon} \cdot \boldsymbol{f} = -\int_{0}^{\infty} dp \int_{-a}^{a} dq\,\tilde{r}^{*}\tilde{r}\delta(p - \varepsilon)\,\frac{\bar{h}^{\sharp}(p,q)}{4\pi\tilde{r}^{*}\tilde{r}^{3}}$$

$$= -\frac{1}{2} \int_{-a}^{a} dq\,\frac{\bar{h}^{\sharp}(\varepsilon,q)}{(\varepsilon + iq)^{2}}\,.$$

As claimed, $\boldsymbol{J}_{\varepsilon}(\boldsymbol{z})$ can be represented by a smooth (but complex) surface current density on $\mathcal{D}_{\varepsilon}(\boldsymbol{y})$. To consider $\varepsilon \to 0$, subtract and add a linear Taylor polynomial in the numerator:

$$\langle\, \boldsymbol{J}_{\varepsilon}\,,\boldsymbol{f}\,\rangle = -\frac{1}{2} \int_{-a}^{a} dq\,\frac{\bar{h}^{\sharp}(\varepsilon,q) - \bar{h}^{\sharp}(\varepsilon,0) - q\partial_{q}\bar{h}^{\sharp}(\varepsilon,0)}{(\varepsilon + iq)^{2}}$$

$$-\frac{1}{2} \int_{-a}^{a} dq\,\frac{\bar{h}^{\sharp}(\varepsilon,0) + q\partial_{q}\bar{h}^{\sharp}(\varepsilon,0)}{(\varepsilon + iq)^{2}}\,.$$

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Again, the smoothness of h on $\mathcal{D}$ implies

$$\bar{h}^\sharp(0, -q) = \bar{h}^\sharp(0, q) \quad \text{and} \quad \partial_q \bar{h}^\sharp(0, 0) = 0.$$

Taking $\varepsilon \to 0$ gives the action of the *true* current distribution:

$$\langle\, \boldsymbol{J}, \boldsymbol{f}\,\rangle = \int_0^a dq\, \frac{\bar{h}^\sharp(0, q) - \bar{h}^\sharp(0, 0)}{q^2} + \frac{1}{a}\,\bar{h}^\sharp(0, 0)\,.$$

This can be expressed in cylindrical coordinates as

$$\boxed{\langle\, \boldsymbol{J}, \boldsymbol{f}\,\rangle = \bar{f}_\varphi^\flat(a, 0) + \int_0^a \varrho\, d\varrho\, \frac{\varrho \bar{f}_\varphi^\flat(\varrho, 0) - a \bar{f}_\varphi^\flat(a, 0)}{(a^2 - \varrho^2)^{3/2}}.} \tag{19}$$

The first term is the mean of $f_\varphi^\flat$ on $\mathcal{R}$, while the second term is a regularization of the surface current density $\boldsymbol{K}$ in (5), taking into account the singularity on $\mathcal{R}$.

Similar remarks apply to those made earlier on ρ:
• $\boldsymbol{J}$ is *real*, so there are no magnetic currents: $\nabla \times \boldsymbol{E} \equiv \boldsymbol{0}$.
• The subtraction in the second term makes ρ a regularized distribution in the sense of [GS64].
• If $\boldsymbol{f} = \boldsymbol{0}$ on $\mathcal{R}$, (19) reduces to

$$\langle\, \boldsymbol{J}, \boldsymbol{f}\,\rangle = \frac{1}{2\pi} \int_0^a \varrho\, d\varrho \int_0^{2\pi} d\varphi\, \frac{\varrho\, \boldsymbol{e}_\varphi \cdot \boldsymbol{f}}{(a^2 - \varrho^2)^{3/2}}\,,$$

reproducing the surface current density (5).
• Unlike the singular expression (5) for $\boldsymbol{K}$, (19) has the correct point-source limit $\boldsymbol{y} \to \boldsymbol{0}$:

$$\langle\, \boldsymbol{J}, \boldsymbol{f}\,\rangle \to \bar{f}_\varphi^\flat(0, 0) = 0, \quad \text{hence} \quad \boldsymbol{J}(\boldsymbol{x} + i\boldsymbol{y}) \to \boldsymbol{0}.$$

5. The Magnetic Moment

The circulating current $\boldsymbol{J}(\boldsymbol{z})$ generates a magnetic moment

$$\boldsymbol{\mu} = \frac{1}{2} \int_{\mathbb{R}^3} \boldsymbol{x} \times \boldsymbol{J}(\boldsymbol{x} + i\boldsymbol{y})\, d\boldsymbol{x}.$$

Using (17) and (18) gives

$$\boldsymbol{x} \times \boldsymbol{J}_\varepsilon = \frac{\delta(p - \varepsilon)\, \boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{y})}{4\pi \tilde{r}^* \tilde{r}^3} = \frac{\delta(p - \varepsilon)\, (az\boldsymbol{x} - r^2\boldsymbol{y})}{4\pi \tilde{r}^* \tilde{r}^3}$$
$$= \frac{\delta(p - \varepsilon)\, (az\boldsymbol{\varrho} + z^2\boldsymbol{y} - r^2\boldsymbol{y})}{4\pi \tilde{r}^* \tilde{r}^3} = \frac{\delta(p - \varepsilon)\, (az\boldsymbol{\varrho} - \varrho^2\boldsymbol{y})}{4\pi \tilde{r}^* \tilde{r}^3}.$$

By (11) and (15), we have

$$\boldsymbol{\mu}_\varepsilon \equiv \frac{1}{2} \int_{\mathbb{R}^3} \boldsymbol{x} \times \boldsymbol{J}_\varepsilon \, d\boldsymbol{x} = \frac{1}{2a} \int_{-a}^{a} \frac{dq}{4\pi(\varepsilon + iq)^2} \int_0^{2\pi} d\varphi \, (az\boldsymbol{\varrho} - \varrho^2\boldsymbol{y})$$
$$= -\frac{(a^2 + \varepsilon^2)\hat{\boldsymbol{y}}}{4a^2} \int_{-a}^{a} dq \, \frac{a^2 - q^2}{(\varepsilon + iq)^2}.$$

Using (12) and letting $\varepsilon \to 0$ gives

$$\boldsymbol{\mu} \equiv \lim_{\varepsilon \to 0} \boldsymbol{\mu}_\varepsilon = -a\hat{\boldsymbol{y}} = -\boldsymbol{y}.$$

The magnetic moment of a charge e displaced to $+i\boldsymbol{y}$ (instead of $-i\boldsymbol{y}$) is therefore

$$\boxed{\boldsymbol{\mu} \equiv \frac{1}{2} \int_{\mathbb{R}^3} \boldsymbol{x} \times \boldsymbol{J}(\boldsymbol{x} - i\boldsymbol{y}) \, d\boldsymbol{x} = ec\,\boldsymbol{y}.} \tag{20}$$

6. A "Newtonian" Gravitomagnetic Field?

In Einstein's theory, a rotating mass generates a *gravitomagnetic field* analogous to the magnetic field generated by a rotating charge [TPM86, CW95]. This manifests itself by "dragging" inertial frames near the body along the direction of motion.

We now repeat the above analysis with the Coulomb potential replaced by Newton's potential and attempt to interpret the results in terms of a "Newtonian" gravitomagnetic field.

Although the gravitomagnetic field has no strict counterpart in Newtonian theory, recall that the rim $\mathcal{R}$ of $\mathcal{D}$ moves at the speed of light, so our situation is essentially relativistic. In any case, it is interesting to follow the mathematics even if its physical significance is as yet unclear.

Begin by replacing Coulomb's potential with Newton's:

$$\phi_e(\boldsymbol{x}) = \frac{e}{4\pi r} \quad \to \quad \phi_m(\boldsymbol{x}) = -\frac{m}{4\pi r}.$$

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The holomorphic Newtonian potential and force field are

$$\tilde{\phi}_m(\boldsymbol{z}) = -\frac{m}{4\pi\tilde{r}}, \quad \tilde{\boldsymbol{F}}(\boldsymbol{z}) = -\nabla\tilde{\phi}_m(\boldsymbol{z}) = -\frac{m\boldsymbol{z}}{\tilde{r}^3}.$$

We want to investigate the physical significance of the real fields

$$\boldsymbol{F}(\boldsymbol{z}) = \text{Re } \tilde{\boldsymbol{F}}(\boldsymbol{z}), \quad \boldsymbol{G}(\boldsymbol{z}) = \text{Im } \tilde{\boldsymbol{F}}(\boldsymbol{z}),$$

which are discontinuous on $\mathcal{D}(\boldsymbol{y})$ and singular on $\mathcal{R}(\boldsymbol{y})$.

By our earlier argument, we have the gravitational counterpart of the homogeneous Maxwell equations:

$$\nabla \times \boldsymbol{F}(\boldsymbol{x} + i\boldsymbol{y}) \equiv \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{G}(\boldsymbol{x} + i\boldsymbol{y}) \equiv 0 \qquad \forall \boldsymbol{x} \in \mathbb{R}^3.$$

• The conservative field $\boldsymbol{F}(\boldsymbol{x}+i\boldsymbol{y})$ will be interpreted as the Newtonian force field due to a mass distribution on $\mathcal{D}(\boldsymbol{y})$.

• The divergenceless field $\boldsymbol{G}(\boldsymbol{x} + i\boldsymbol{y})$ is, by definition, the "Newtonian" gravitomagnetic field.

We define the *mass density* and *mass current density* by

$$\rho_m(\boldsymbol{z}) \equiv -\nabla \cdot \boldsymbol{F}(\boldsymbol{z}), \qquad \boldsymbol{J}_m(\boldsymbol{z}) \equiv -\nabla \times \boldsymbol{G}(\boldsymbol{z}),$$

where the sign is chosen so that $\rho_m(\boldsymbol{x}) = m\delta(\boldsymbol{x})$ in the case of a real point source.

The electromagnetic and gravitational cases are related by

$$\boldsymbol{F}(\boldsymbol{z}) = -(m/e)\boldsymbol{E}(\boldsymbol{z}), \qquad \boldsymbol{G}(\boldsymbol{z}) = -(m/e)\boldsymbol{H}(\boldsymbol{z})$$
$$\rho_m(\boldsymbol{z}) = (m/e)\rho_e(\boldsymbol{z}), \qquad \boldsymbol{J}_m(\boldsymbol{z}) = (m/e)\boldsymbol{J}_e(\boldsymbol{z}).$$

These relations can be used to transcribe all our results to the gravitational case.

In particular, placing a point mass m at $i\boldsymbol{y}$ transforms it into a rigidly rotating disk $\mathcal{D}$ with angular velocity $\boldsymbol{\omega} = c\,\hat{\boldsymbol{y}}/a$. By (20), its *spin angular momentum* is

$$\boldsymbol{l} \equiv \int_{\mathbb{R}^3} \boldsymbol{x} \times \boldsymbol{J}_m(\boldsymbol{x} - i\boldsymbol{y}) \, d\boldsymbol{x} = \frac{m}{e} \int_{\mathbb{R}^3} \boldsymbol{x} \times \boldsymbol{J}_e(\boldsymbol{x} - i\boldsymbol{y}) \, d\boldsymbol{x} = \frac{2m}{e}\,\boldsymbol{\mu}.$$

The "Newtonian" gyromagnetic ratio is therefore $\gamma_N = e/2m$, the classical value for a distribution of charged matter with uniform charge-to mass density ratio.

This differs markedly from the gyromagnetic ratio of a Kerr-Newman solution [DKS69], which has the Dirac value $\gamma = e/m$.

7. Why do Imaginary Translations Generate Spin?

We have seen that the *formal* imaginary translation of a point charge from the origin to $i\boldsymbol{y}$ has *two real, physical effects:*

- The point singularity **opens** to sweep out an oriented disk $\mathcal{D}$.
- $\mathcal{D}$ inevitably comes with its *maximum allowed spin*, rotating rigidly at the angular velocity $\boldsymbol{\omega} = (c/a)\,\hat{\boldsymbol{y}}$ such that the rim $\mathcal{R}$ moves at the speed of light.
- The original charge is distributed uniformly on $\mathcal{R} = \partial\mathcal{D}$, while additional charges $Q_{\mathcal{R}} = \infty$ and $Q_{\mathcal{D}} = -\infty$ with zero sum are distributed over $\mathcal{R}$ (uniformly) and $\mathcal{D}$ (with surface density σ).

How is this magic to be understood? Already in the 1950s, Ivor Robinson was advocating the use of complex spacetime in general relativity and electrodynamics, together with *self-dual fields* of the type $\boldsymbol{E} + i\boldsymbol{H}$; see the Introduction by Rindler and Trautman in [RT87]; also [T62, S58].

This led in 1961 to the discovery [R01] of what Penrose later called the *Robinson congruence,* instrumental in the formulation of Twistor theory; see [P67] and *On the Origins of Twistor Theory* by Roger Penrose in [RT87].

After Kerr's landmark paper on the gravitational field of a spinning mass [K63], Newman and his collaborators showed that the Kerr metric and its charged counterpart (Kerr-Newman metric) can be "derived" from the Schwarzschild metric using a complex spacetime coordinate transformation [NJ65, N65].

Newman et al. put "derive" in quotes because they were at the time unable to explain *why* these transformations should lead to another solution of the Einstein equation.

Much work has been done since then to develop and clarify this idea [NW74, ABS75, F76]. One of the most beautiful explanations of the connection between complex translations in *flat* spacetime and the generation of spin is given by Newman and Winicour in [NW74a], but it seems to me that not all of the mystery has been explained regarding the success of this method in *curved* spacetime.

I have pursued complex spacetime in a very different direction, as a *relativistic phase space* on which to build quantum physics, unaware at the time

of all the above work [K90, K94].

There are reasons to hope that the analysis of the disk singularity given here may be extended to curved spacetime, leading to *distributional energy-momentum tensors* as sources for the Kerr and Kerr-Newman fields.

It should also be very useful to extend the above analysis from static to radiating fields, in particular to the *electromagnetic pulsed-beam wavelets* proposed for applications to radar and communications [K00a].

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